Gaussian variable and scalar field

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Abstract

Here, I list mathematical formulations of Gaussian theories and a scalar field theory with anharmonic fourth order interactions [1]. This is for my own reference but might also be useful for someone else. All materials here are copied from the Chapter I.7 of A. Zee's QFT book with some typos corrected [1].

1. Univariate Gaussian

We want to evaluate the following integral

$$Z(J,\lambda) = \int_{-\infty}^{+\infty} dq\ e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq}$$

1.1 Expansion with respect to $\boldsymbol{\lambda}$

$$\begin{aligned}
Z(J,\lambda) &= \int_{-\infty}^{+\infty} dq\ e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq} \\
&= \int_{-\infty}^{+\infty} dq\ e^{-\frac{1}{2}m^2q^2 + Jq}\left[1 - \frac{\lambda}{4!}q^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 q^8 + \cdots\right] \\
&= \left[1 - \frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8 + \cdots\right]\int_{-\infty}^{+\infty} dq\ e^{-\frac{1}{2}m^2q^2 + Jq} \\
&= e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4}\int_{-\infty}^{+\infty} dq\ e^{-\frac{1}{2}m^2q^2 + Jq} \\
&= \left(\frac{2\pi}{m^2}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4} e^{\frac{1}{2m^2}J^2} \\
&\equiv Z(0,0) e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4} e^{\frac{1}{2m^2}J^2}
\end{aligned}$$

where $Z(0,0) \equiv \left(\frac{2\pi}{m^2}\right)^{\frac{1}{2}}$.

1.2 Expansion with respect to $\boldsymbol{J}$

$$Z(J,\lambda) = \int_{-\infty}^{+\infty} dq\ e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq}$$

$$= \sum_{s=0}^{\infty} \frac{1}{s!} J^s \int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4} q^s$$

$$\equiv Z(0,0) \sum_{s=0}^{\infty} \frac{1}{s!} J^s G^{(s)}$$

where $G^{(s)} = \frac{1}{Z(0,0)} \int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4} q^s$.

2. Multivariate Gaussian

Similar to the univariate case, we evaluate the following integral

$$Z(J,\lambda) = \int_{-\infty}^{+\infty} dq_1 dq_2 \cdots dq_N\, e^{-\frac{1}{2}q\cdot A\cdot q - \frac{\lambda}{4!}q^4 + J\cdot q}$$

$$= \int_{-\infty}^{+\infty} \left(\prod_l dq_l\right) e^{-\frac{1}{2}q\cdot A\cdot q - \frac{\lambda}{4!}q^4 + J\cdot q}$$

where $q^4 \equiv \sum_i q_i^4$.

2.1 Expansion with respect to λ

$$Z(J,\lambda) = \int_{-\infty}^{+\infty} \left(\prod_l dq_l\right) e^{-\frac{1}{2}q\cdot A\cdot q - \frac{\lambda}{4!}q^4 + J\cdot q}$$

$$= \left[\frac{(2\pi)^N}{\det[A]}\right]^{\frac{1}{2}} e^{-\frac{\lambda}{4!}\sum_i\left(\frac{\partial}{\partial J_i}\right)^4} e^{\frac{1}{2}J\cdot A^{-1}\cdot J}$$

$$\equiv Z(0,0) e^{-\frac{\lambda}{4!}\sum_i\left(\frac{\partial}{\partial J_i}\right)^4} e^{\frac{1}{2}J\cdot A^{-1}\cdot J}$$

where $Z(0,0) \equiv \left[\frac{(2\pi)^N}{\det[A]}\right]^{\frac{1}{2}}$.

2.2 Expansion with respect to J

$$Z(J,\lambda) = \int_{-\infty}^{+\infty} \left(\prod_l dq_l\right) e^{-\frac{1}{2}q\cdot A\cdot q - \frac{\lambda}{4!}q^4 + J\cdot q}$$

$$= \sum_{i_1\cdots i_s} \sum_{s=0}^{\infty} \frac{1}{s!} J_{i_1} \cdots J_{i_s} \int_{-\infty}^{+\infty} \left(\prod_l dq_l\right) e^{-\frac{1}{2}q\cdot A\cdot q - \frac{\lambda}{4!}q^4} q_{i_1} \cdots q_{i_s}$$

$$\equiv Z(0,0) \sum_{i_1\cdots i_s} \sum_{s=0}^{\infty} \frac{1}{s!} J_{i_1} \cdots J_{i_s} G^{(s)}_{i_1\cdots i_s}$$

where $G^{(s)}_{i_1\cdots i_s} = \frac{1}{Z(0,0)}\int_{-\infty}^{+\infty}(\prod_l dq_l)\, e^{-\frac{1}{2}q\cdot A\cdot q-\frac{\lambda}{4!}q^4} q_{i_1}\cdots q_{i_s}$.

3. Scalar field theory

Now, we deal with the scalar field theory with the following integral

$$Z(J,\lambda) = \int D\varphi e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]-\frac{\lambda}{4!}\varphi^4+J\varphi\right\}}$$

where $J = J(x), \varphi = \varphi(x)$.

3.1 Expansion with respect to λ (The Schwinger way)

$$\begin{aligned} Z(J,\lambda) &= \int D\varphi e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]-\frac{\lambda}{4!}\varphi^4+J\varphi\right\}} \\ &= e^{-\frac{i}{4!}\lambda\int d^4w\left[\frac{\delta}{i\delta J(w)}\right]^4}\int D\varphi e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]+J\varphi\right\}} \\ &= Z(0,0)e^{-\frac{i}{4!}\lambda\int d^4w\left[\frac{\delta}{i\delta J(w)}\right]^4} e^{-\frac{i}{2}\int d^4x d^4y J(x)D(x-y)J(y)} \end{aligned}$$

where the propagator $D(x-y)$ plays the role of $(-\frac{1}{m^2})$ in the univariate case and the role of A^{-1} in the multivariate case, and can be written as a Fourier expansion

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-y)}}{k^2-m^2+i\epsilon}$$

where ϵ is some small positive number and will be set to zero when there is no danger of singularity. The propagator is the solution of the following equation

$$-(\partial^2+m^2)D(x-y) = \delta^4(x-y)$$

3.2 Expansion with respect to J (The Wick way)

$$\begin{aligned} Z(J,\lambda) &= \int D\varphi e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]-\frac{\lambda}{4!}\varphi^4+J\varphi\right\}} \\ &= \int d^4x_1\cdots d^4x_s \sum_{s=0}^{\infty}\frac{1}{s!}J(x_1)\cdots J(x_s)\int D\varphi e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]-\frac{\lambda}{4!}\varphi^4\right\}}\varphi(x_1)\cdots\varphi(x_s) \\ &\equiv Z(0,0)\int d^4x_1\cdots d^4x_s\sum_{s=0}^{\infty}\frac{1}{s!}J(x_1)\cdots J(x_s)G^{(s)}(x_1,\cdots,x_s) \end{aligned}$$

where $G^{(s)}(x_1,\cdots,x_s) \equiv \frac{1}{Z(0,0)}\int D\varphi e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]-\frac{\lambda}{4!}\varphi^4\right\}}\varphi(x_1)\cdots\varphi(x_s)$.

Reference

[1] Zee, A., 2010. *Quantum field theory in a nutshell* (Vol. 7). Princeton university press.